

# Spectral extrema and Lifshitz tails for non monotonous alloy type models

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## Abstract

In the present note, we determine the ground state energy and study the existence of Lifshitz tails near this energy for some non monotonous alloy type models. Here, non monotonous means that the single site potential coming into the alloy random potential changes sign. In particular, the random operator is not a monotonous function of the random variables.

RÉSUMÉ. Cet article est consacré à la détermination de l'énergie de l'état fondamental et à l'étude de possibles asymptotiques de Lifshitz au voisinage de cette énergie pour certains modèles d'Anderson continus non monotones. Ici, non monotone signifie que le potentiel de simple site entrant dans la composition du potentiel aléatoire change de signe. En particulier, l'opérateur aléatoire n'est pas une fonction monotone des variables aléatoires.

## 0 Introduction and results

In this paper, we consider the continuous alloy type (or Anderson) random Schrödinger operator:

$$H_\omega = -\Delta + V_\omega \text{ where } V_\omega(x) = \sum_{\gamma \in \mathbb{Z}^d} \omega_\gamma V(x - \gamma) \quad (0.1)$$

on  $\mathbb{R}^d$ ,  $d \geq 1$ , where  $V$  is the site potential, and  $(\omega_\gamma)_{\gamma \in \mathbb{Z}^d}$  are the random coupling constants. Throughout this paper, we assume

**(H1)** (1)  $V : \mathbb{R}^d \rightarrow \mathbb{R}$  is  $L^p$  (where  $p = 2$  if  $d \leq 3$  and  $p > d/2$  if  $d > 3$ ), non identically vanishing and supported in  $(-1/2, 1/2)^d$ ;

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- (2)  $(\omega_\gamma)_\gamma$  are independent identically distributed (i.i.d.) random variables distributed in  $[a, b]$  ( $a < b$ ) with essential infimum  $a$  and essential supremum  $b$ .

Let  $\Sigma$  be the almost sure spectrum of  $H_\omega$  and  $E_- = \inf \Sigma$ . When  $V$  has a fixed sign, it is well known that the  $E_- = \inf(\sigma(-\Delta + V_b))$  if  $V \leq 0$  and  $E_- = \inf(\sigma(-\Delta + V_a))$  if  $V \geq 0$ . Here,  $\bar{x}$  is the constant vector  $\bar{x} = (x)_{\gamma \in \mathbb{Z}^d}$ . Moreover, in this case, it is well known that the integrated density of states of the Hamiltonian (see e.g. (0.3)) admits a Lifshitz tail near  $E_-$ , i.e., that the integrated density of states at energy  $E$  decays exponentially fast as  $E$  goes to  $E_-$  from above. We refer to [9, 7, 22, 20, 6, 5, 11] for precise statements.

In the present paper, we address the case when  $V$  changes sign, i.e., there may exist  $x_+ \neq x_-$  such that

$$V(x_-) \cdot V(x_+) < 0. \quad (0.2)$$

The basic difficulty this property introduces is that the variations of the potential  $V_\omega$  as a function of  $\omega$  are not monotonous. In the monotonous case, to get the minimum, one can simply minimize with respect to each of the random variables individually. In the non monotonous case, this uncoupling between the different random variables may fail. Our results concern reflection symmetric potentials since, as we will see, for these potentials we also have an analogous decoupling between the different random variables. Thus, we make the following symmetry assumption on  $V$ :

**(H2)**  $V$  is reflection symmetric i.e. for any  $\sigma = (\sigma_1, \dots, \sigma_d) \in \{0, 1\}^d$  and any  $x = (x_1, \dots, x_d) \in \mathbb{R}^d$ ,

$$V(x_1, \dots, x_d) = V((-1)^{\sigma_1} x_1, \dots, (-1)^{\sigma_d} x_d).$$

We now consider the operator  $H_\lambda^N = -\Delta + \lambda V$  with Neumann boundary conditions on the cube  $[-1/2, 1/2]^d$ . Its spectrum is discrete, and we let  $E_-(\lambda)$  be its ground state energy. It is a simple eigenvalue and  $\lambda \mapsto E_-(\lambda)$  is a real analytic concave function defined on  $\mathbb{R}$ . We first observe:

**Proposition 0.1.** *Under the above assumptions (H1) and (H2),*

$$E_- = \inf(E_-(a), E_-(b)).$$

For  $a$  and  $b$  sufficiently small, this result was proven in [17] without the assumption (H2) but with an additional assumption on the sign of  $\int_{\mathbb{R}^d} V(x) dx$ . The method used by Najar relies on a small coupling constant expansion for the infimum of  $\Sigma$ . These ideas were first used in [3] to treat other non

monotonous perturbations, in this case magnetic ones, of the Laplace operator. In [1], the authors study the minimum of the almost sure spectrum for a random displacement model i.e. the random potential is defined as  $V_\omega(x) = \sum_{\gamma \in \mathbb{Z}^d} V(x - \gamma - \xi_\gamma)$  where  $(\xi_\gamma)_\gamma$  are i.i.d. random variables supported in a sufficiently small compact.

We now turn to the results on Lifshitz tails. We denote by  $N(E)$  the integrated density of states of  $H_\omega$ , i.e., it is defined by the limit

$$N(E) = \lim_{L \rightarrow +\infty} \frac{\#\{\text{eigenvalues of } H_{\omega,L}^N \leq E\}}{(2L+1)^d} \quad (0.3)$$

where  $H_{\omega,L}^N$  is the operator  $H_\omega$  restricted to the cube  $[-L-1/2, L+1/2]^d$  with Neumann boundary conditions. This limit exists for a.s.  $\omega$  and is independent of  $\omega$ ; it has been the object of a lot of studies and we refer to [20, 23, 24] for extensive reviews.

We first give an upper bound on the integrated density of states. In the applications of the Lifshitz tails asymptotics, in particular, to localization, this side of the bound is the most important and also the difficult one to obtain.

**Theorem 0.1.** *Suppose assumptions (H1) and (H2) are satisfied. Assume moreover that*

$$E_-(a) \neq E_-(b). \quad (0.4)$$

*Then*

$$\limsup_{E \rightarrow E_-^+} \frac{\log |\log N(E)|}{\log(E - E_-)} \leq -\frac{d}{2} - \alpha_+ \quad (0.5)$$

*where we have set  $c = a$  if  $E_-(a) < E_-(b)$  and  $c = b$  if  $E_-(a) > E_-(b)$ , and*

$$\alpha_+ = -\frac{1}{2} \liminf_{\varepsilon \rightarrow 0} \frac{\log |\log \mathbb{P}(\{|c - \omega_0| \leq \varepsilon\})|}{\log \varepsilon} \geq 0.$$

As will be clear from the proofs, we could also consider the model  $H_\omega = H_0 + V_\omega$  where  $V_\omega$  is as above and  $H_0 = -\Delta + W$  where  $W$  is a  $\mathbb{Z}^d$ -periodic potential that satisfies the symmetry assumption (H2).

We now study a lower bound for the integrated density of states that we will prove in a more general case than the upper bound, i.e., we don't need to assume (H2). The assumption is as follows:

**(HP)** there exists  $\omega^P \in [a, b]^{\mathbb{Z}^d}$  that is periodic (, i.e., for some  $L_0 \in \mathbb{N}$ , for all  $\gamma \in \mathbb{Z}^d$  and  $\beta \in \mathbb{Z}^d$ ,  $\omega_{\gamma+L_0\beta}^P = \omega_\gamma^P$ ) such that  $\inf \Sigma = \inf \sigma(H_{\omega^P})$ .

Under this assumption, we have

**Theorem 0.2.** *Let  $H_\omega$  be defined as above, and assume (H1) and (HP) hold. Then*

$$-\frac{d}{2} - \alpha_- \leq \liminf_{E \rightarrow E_-^+} \frac{\log |\log N(E)|}{\log(E - E_-)}, \quad (0.6)$$

where

$$\alpha_- = -\liminf_{\varepsilon \rightarrow 0} \frac{\log |\log \mathbb{P}(\{\forall \gamma \in \mathbb{Z}^d / L_0 \mathbb{Z}^d; |\omega_\gamma^P - \omega_\gamma| \leq \varepsilon\})|}{\log \varepsilon}.$$

Assume now that (H1) and (H2) hold, and hence, (HP) holds with  $\omega^P = \bar{a}$  or  $\omega^P = \bar{b}$ . Indeed, as we will see in the proof of Proposition 0.1 in Section 1, under assumption (H2),  $E_-(a)$  is also the bottom of the spectrum of the periodic operator  $H_{\bar{a}}$ . If we assume that  $\alpha_+ = \alpha_- = 0$  and  $E_-(a) \neq E_-(b)$ , we obtain the following corollary:

**Theorem 0.3.** *Assume that (H1) and (H2) and (0.4) hold and that  $\alpha_- = \alpha_+ = 0$  then*

$$\lim_{E \rightarrow E_-^+} \frac{\log |\log N(E)|}{\log(E - E_-)} = -\frac{d}{2}.$$

Combining Theorem 0.1 with the Wegner estimates obtained in [10, 4] and the multiscale analysis as developed in [2], we learn

**Theorem 0.4.** *Assume (H1), (H2) and (0.4) hold. Assume, moreover, that the common distribution of the random variables admits an absolutely continuous density. Then, the bottom edge of the spectrum of  $H_\omega$  exhibits complete localization in the sense of [2].*

Lifshitz tail have already been proved for various non monotonous random models, mainly models with a random magnetic fields (see e.g. [3, 15, 18, 19]). In the models we consider, we will now see that Lifshitz tails do not always appear.

In a companion paper (see [14]), we study the case when  $E_-(a) = E_-(b)$ . This requires techniques different from the ones used in the present paper and gets particularly interesting when the random variables are Bernoulli distributed. However, it is quite easy to see that, when  $E_-(a) = E_-(b)$ , Lifshitz tails may fail; the density of states can even exhibit a van Hove singularity.

**Theorem 0.5.** *There exists potentials  $V$  and random variables  $(\omega_\gamma)_\gamma$  satisfying (H1) and (H2) such that*

- $E_-(a) = E_-(b) = 0$
- *there exists  $C > 0$  such that, for  $E \geq 0$ , one has*

$$\frac{1}{C} E^{d/2} \leq N(E) \leq C E^{d/2}.$$

This paper is constructed as follows. In Section 1, we determine the bottom of the almost sure spectrum, and prove Proposition 0.1. In Section 2,

we prove our main theorems, Theorem 0.2 and Theorem 0.4. Finally, in section 3, we prove Theorem 0.5.

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## 1 Determining the bottom of the spectrum

We denote by  $t \mapsto E_-(t)$  the ground state energy of the operator  $H_{t,0}^N$ , i.e.,  $-\Delta + tV$  on  $[-1/2, 1/2]^d$  with Neumann boundary conditions. We first note that  $E_-(t)$  is a concave function of  $t$  as, by the variational principle, it is the infimum of a family of affine functions of  $t$ . Hence, for  $t \in [a, b]$ , we have  $E_-(t) \geq \min(E_-(a), E_-(b))$ . Then, partitioning  $\mathbb{R}^d$  into the cubes  $\gamma + [-1/2, 1/2]^d$  for  $\gamma \in \mathbb{Z}^d$ , and, restricting  $H_\omega$  to each of these cubes with Neumann boundary conditions, we obtain

$$H_\omega \geq \bigoplus_{\gamma \in \mathbb{Z}^d} H_{\omega_\gamma, 0}^N$$

So, we learn

$$H_\omega \geq \min(E_-(a), E_-(b)).$$

We let  $L \geq 1$ , and consider  $H_{\omega, L}^P$ , the operator  $H_\omega$  restricted to the cube  $[-L - 1/2, L + 1/2]^d$  with periodic boundary conditions. Clearly, this operator depends only on finitely many random variables. We prove

**Lemma 1.1.**

$$\Sigma = \overline{\bigcup_{L \geq 1} \bigcup_{\omega \text{ admissible}} \sigma(H_{\omega, L}^P)},$$

where  $\omega$  is called *admissible* if all the components of  $\omega$  are in the support of the distribution of the random variables defining the alloy type random operator.

This lemma is a variant of a standard characterization of the almost sure spectrum of an alloy type model. To prove Proposition 0.1, i.e., that  $E_- = \min(E_-(a), E_-(b))$ , it is hence enough to prove that, for any  $L$  sufficiently large,

$$\inf_{\omega \in [a, b]^{C_L^d}} \inf \sigma(H_{\omega, L}^P) \leq \min(E_-(a), E_-(b)) \quad (1.1)$$

where  $C_L^d = \mathbb{Z}^d \cap [-L - 1/2, L + 1/2]^d$ . To prove (1.1), we will use the assumption (H.2). For the sake of definiteness, let us assume  $E_-(a) \leq E_-(b)$ .

The ground state of  $H_{a, 0}^N$ , say  $\psi$ , is simple and can be chosen uniquely as a normalized positive function. The reflection symmetry of the potential  $V$  guarantees that  $\psi$  is reflection symmetric. For  $\gamma \in \mathbb{Z}^d$  such that

$|\gamma| = |\gamma_1| + \dots + |\gamma_d| = 1$ , we can continue  $\psi$  to the  $\gamma + [-1/2, 1/2]^d$  by reflection symmetry with respect to the common boundary of  $[-1/2, 1/2]^d$  and  $\gamma + [-1/2, 1/2]^d$ . As  $\psi$  is reflection symmetric, we continue this process of reflection with respect to the boundaries of the new cubes to obtain a continuation of  $\psi$  that is  $\mathbb{Z}^d$ -periodic, positive and reflection symmetric with respect to any plane that is common boundary to two cubes of the form  $\gamma + [-1/2, 1/2]^d$ . Moreover,  $\psi$  satisfies, for any  $L \geq 0$ ,  $H_{a,L}^P \psi = H_{a,0}^P \psi = H_{a,0}^N \psi = E_-(a)\psi$ . This proves that  $E_-(a) \geq \inf \sigma(H_{a,L}^P)$ . Hence, (1.1) holds. This completes the proof of Proposition 0.1.  $\square$

*Proof of Lemma 1.1.* Recall a well known characterization of the almost sure spectrum of an alloy type model in terms of periodic approximations (see e.g. [20, 5]). Therefore, for  $L \geq 1$ , define the  $L\mathbb{Z}^d$ -periodic operator

$$H_{\omega,L} = -\Delta + V_{\omega,L}, \quad V_{\omega,L}(\cdot) = \sum_{\beta \in L\mathbb{Z}^d} \sum_{\gamma \in \mathbb{Z}^d / (L\mathbb{Z}^d)} \omega_\gamma V(\cdot - \beta - \gamma). \quad (1.2)$$

Then, one has

$$\Sigma = \overline{\bigcup_{L \geq 1} \bigcup_{(\omega_\gamma)_{\gamma \in \mathbb{Z}^d / (L\mathbb{Z}^d)} \text{ admissible}} \sigma(H_{\omega,L})}$$

where  $(\omega_\gamma)_{\gamma \in \mathbb{Z}^d / (L\mathbb{Z}^d)}$  is admissible if all its components belong to the support of the random variables defining the alloy type model.

Floquet theory (see e.g. [21]) guarantees that  $\sigma(H_{\omega,L}^P) \subset \sigma(H_{\omega,L})$ . So, in order to prove Lemma 1.1, it is sufficient to prove that

$$\sigma(H_{\omega,L}) \subset \overline{\bigcup_{n \geq 1} \sigma(H_{\omega^L, nL}^P)} \quad (1.3)$$

for some well chosen admissible  $\omega^L$ .

Consider  $\omega^L$  defined by  $\omega_{\gamma+L\beta}^L = \omega_\gamma$  for  $\gamma \in \mathbb{Z}^d / (L\mathbb{Z}^d)$  and  $\beta \in \mathbb{Z}^d$ . Clearly,  $V_{\omega^L} = V_{\omega,L}$ ; hence, if  $E_n(\theta)$  are the Floquet eigenvalues of  $H_{\omega,L}$ , the spectrum of the operator  $H_{\omega^L, nL}^P$  is the set  $\{E_n(2\pi\gamma/n); \gamma \in \mathbb{Z}^d / (nL\mathbb{Z}^d)\}$  (see e.g. [13]). The inclusion (1.3) follows from the continuity of the Floquet eigenvalues as function of the Floquet parameter (see e.g Lemma 7.1 in [16]).  $\square$

## 2 Lifshitz tails

To fix ideas let us assume  $E_-(a) < E_-(b)$ . The two bounds in Theorem 0.1 and Theorem 0.2 will be proved separately.

## 2.1 The upper bound

The upper bound on the integrated density of states, Theorem 0.1, will be an immediate consequence of the following result.

**Theorem 2.1.** *Suppose assumptions (H1) and (H2) are satisfied, and, that  $E_-(a) < E_-(b)$ . Then, there exists  $c > 0$  such that, for  $E \geq E_-(a)$ , one has*

$$N(E) \leq N_m(C(E - E_-(a))) \quad (2.1)$$

where  $N_m$  is the integrated density of states of the random operator

$$H_\omega^m = H_{\bar{a}} - E_-(a) + \sum_{\gamma \in \mathbb{Z}^d} (\omega_\gamma - a) \mathbf{1}_{[-1/2, 1/2]^d}(x - \gamma) \quad (2.2)$$

and  $H_{\bar{a}}$  is defined above.

The upper bound is then deduced from the same bound for the integrated density of states of  $H_\omega^m$  which is standard, see e.g. [20, 5, 22] and references therein.

*Proof.* We first note it is well known that, at  $E$ , a continuity point of  $N(E)$ , the sequence

$$N_L^N(E) = \mathbb{E} \left( \frac{\#\{\text{eigenvalues of } H_{\omega, L}^N \leq E\}}{(2L+1)^d} \right) \quad (2.3)$$

is decreasing and converges to  $N(E)$  (see e.g. [20, 5]). So to prove Theorem 2.1, it suffices to prove that, there exists  $C > 0$  such that, for  $E$  real and  $L$  large

$$N_L^N(E) \leq N_L^m(C(E - E_-(a)))$$

where  $N_L^m(E)$  is defined by (2.3) where  $H_{\omega, L}^N$  is replaced by  $H_{\omega, L}^m$ , i.e., the restriction of  $H_\omega^m$  to  $[-L-1/2, L+1/2]^d$  with Neumann boundary conditions. By the Rayleigh-Ritz principle ([21], Section XIII.1), this follows from the quadratic form inequality

$$H_{\omega, L}^m \leq C(H_{\omega, L}^N - E_-(a)).$$

Under our assumptions on  $V$ , the form domain of both of these operators is  $H^1([-L-1/2, L+1/2]^d)$ . Now, as for  $\psi \in H^1([-L-1/2, L+1/2]^d)$  and  $\gamma \in \mathbb{Z}^d \cap [-L-1/2, L+1/2]^d$ ,  $\psi \mathbf{1}_{\gamma+[-1/2, 1/2]^d} \in H^1(\gamma + [-1/2, 1/2]^d)$ , this inequality in turns follows from the inequalities

$$\begin{aligned} \forall \gamma \in \mathbb{Z}^d \cap [-L-1/2, L+1/2]^d, \forall \psi \in H^1(\gamma + [-1/2, 1/2]^d), \\ \langle H_{\omega, L}^m \psi, \psi \rangle_{\gamma+[-1/2, 1/2]^d} \leq C \langle (H_{\omega, L}^N - E_-(a)) \psi, \psi \rangle_{\gamma+[-1/2, 1/2]^d}. \end{aligned}$$

Note that, here, the choice of the boundary condition is crucial: the form domain of the Neumann operator is the whole  $H^1$ -space; moreover, the Neumann quadratic form does not involve boundary terms. Taking into account the structure of our random potentials, we see that this will follow from the operator inequality

$$(H_{a,0}^N - E_-(a)) + (t - a) \leq C(H_{t,0}^N - E_-(a)), \quad t \in [a, b], \quad (2.4)$$

where  $H_{t,0}^N = -\Delta + tV$  on  $[-1/2, 1/2]^d$  with Neumann boundary conditions as before.

**Lemma 2.1.** *Let  $H_0$  be self-adjoint on  $\mathcal{H}$  a separable Hilbert space such that  $0 = \inf \sigma(H_0)$ . Let  $V_1$  be a closed symmetric operator relatively bounded with respect to  $H_0$  with bound 0. Set  $H_1 = H_0 + V_1$  and  $E_1 = \inf \sigma(H_1)$ . Assume  $E_1 > 0$ . Then, there exists  $C > 0$  such that, for  $t \in [0, 1]$ , one has*

$$C(H_0 + tV_1) \geq H_0 + t.$$

Lemma 2.1 applies to our case and, as a result, we obtain (2.4). This completes the proof of Theorem 2.1.  $\square$

*Proof of Lemma 2.1.* Regular perturbation theory ensures that, for some  $\beta > 1$ ,  $\inf \sigma(H_0 + \beta V_1) = \delta > 0$ . Then, for  $t \in [0, 1]$ , we have

$$\begin{aligned} H_0 + tV_1 &= (1 - t/\beta)H_0 + (t/\beta)(H_0 + \beta V_1) \\ &\geq (1 - 1/\beta)H_0 + (t/\beta)\delta \geq \frac{1}{C}(H_0 + t) \end{aligned}$$

where  $C^{-1} = \min(1 - 1/\beta, \delta/\beta)$ .  $\square$

## 2.2 The lower bound

We will use the techniques set up in [12, 16]. We first recall two results from these papers which are valid in the generality of the present work. Consider a random Schrödinger operator of the form (0.1) where

- (HL1)  $V$  is a not identically vanishing, real valued, compactly supported function that is in  $L^p$  (where  $p = 2$  if  $d \leq 3$  and  $p > d/2$  if  $d > 3$ );
- (HL2) the random variables  $(\omega_\gamma)_{\gamma \in \Gamma}$  are independent, identically distributed, non trivial and bounded.

Clearly these two assumptions are consequences of assumption (H1). The existence of  $N(E)$ , the integrated density of states defined by (0.3) is known ([20, 23]).



**Theorem 2.2** ([12]). *Assume (HL1) and (HL2) hold. Pick  $\eta_0 > 0$  and  $I \subset \mathbb{R}$ , a compact interval. Then, there exists  $\nu_0 > 0$  and  $\varepsilon_0 > 0$  such that, for  $0 < \varepsilon < \varepsilon_0$ ,  $E_- \in I$  and  $n \geq \varepsilon^{-\nu_0}$ , we have*

$$\begin{aligned} & \mathbb{E}(N_{\omega,n}(E_- + \varepsilon/2) - N_{\omega,n}(E_- - \varepsilon/2)) - e^{-(n\varepsilon^{\nu_0})^{-\eta_0}} \\ & \leq N(E_- + \varepsilon) - N(E_- - \varepsilon) \\ & \leq \mathbb{E}(N_{\omega,n}(E_- + 2\varepsilon) - N_{\omega,n}(E_- - 2\varepsilon)) + e^{-(n\varepsilon^{\nu_0})^{-\eta_0}}. \end{aligned} \quad (2.5)$$

where  $N_{\omega,n}$  is the integrated density of states of the periodic operator  $H_{\omega,n}$  defined in (1.2).

In [12], Theorem 2.2 is not stated in exactly the same form and under slightly stronger but unnecessary assumptions. The modifications necessary to obtain the form given here are simple and left to the reader.

The second result we use is the following

**Lemma 2.2** ([12]). *Consider a periodic Schrödinger operator of the form  $H_{\omega^P}$  where  $\omega^P$  satisfies (HP). Let  $E_- = \inf \sigma(H_{\omega^P})$ . Then, for any  $\alpha \in (0, 1)$  and any  $\varepsilon$  sufficiently small, there exists a function  $w_\varepsilon$  with the following properties*

- (1)  $w_\varepsilon$  is supported in a ball of center 0 and radius  $2\varepsilon^{-(1+2\alpha)/2}$ ;
- (2)  $1 \leq \|w_\varepsilon\|_{L^2}^2 \leq 2$ ;
- (3)  $\|(H_{\omega^P} - E_-)w_\varepsilon\|_{L^2}^2 \leq C\varepsilon^{2(1+\alpha)}$  for some  $C > 0$  (independent of  $\varepsilon$ ).

Though not formulated as a lemma, this result is proved in section 2 of [12].

We now prove the lower bound (0.6). Pick  $\alpha \in (0, 1)$  arbitrary. Let  $\Gamma'$  be the lattice with respect to which  $\omega^P$  is periodic. Pick  $n \in \mathbb{N}^*$  such that  $n \sim \varepsilon^{-\nu}$  for  $\nu > \nu_0$  ( $\nu_0$  given by Theorem 2.2),  $\nu$  to be chosen sufficiently large. As  $w_\varepsilon$  constructed in Lemma 2.2 has compact support in the interior of  $C(0, n)$ , it can be “periodized” to satisfy quasi-periodic boundary conditions; for  $\theta \in \mathbb{R}^d$ , we set

$$w_{\varepsilon,\theta}(\cdot) = \sum_{\beta \in (2n+1)\Gamma'} e^{-i\beta\theta} w_\varepsilon(\cdot + \beta).$$

Then,  $w_{\varepsilon,\theta}$  satisfies  $w_{\varepsilon,\theta}(x + \beta) = e^{i\beta\theta} w_{\varepsilon,\theta}(x)$  for  $x \in \mathbb{R}^d$  and  $\beta \in (2n+1)\Gamma'$ , and we have

$$\|w_{\varepsilon,\theta}\|_{L^2(C(0,n))}^2 \geq 1. \quad (2.6)$$

We define

$$\Lambda_\alpha(\varepsilon) = \left\{ \gamma \in \Gamma'; \text{ for } 1 \leq j \leq d, |\gamma_j| \leq \varepsilon^{-(1+3\alpha)/2} \right\}.$$

Since  $V$  satisfies (HL1), if we assume  $\omega_\gamma \leq \varepsilon^{1+\alpha}$  for  $\gamma \in \Lambda_\alpha(\varepsilon)$ , then

$$\|V_{\omega,n}(H_{\omega^P} - E_- - 1)^{-1}\| \leq C\varepsilon^{1+\alpha}.$$

This, (2.6) and point (3) of Lemma 2.2 imply that, for some  $C > 0$  and  $\varepsilon$  sufficiently small, we have

$$\|(H_{\omega,n,\theta} - E_-)w_{\varepsilon,\theta}\| \leq C\varepsilon^{1+\alpha}\|w_{\varepsilon,\theta}\|.$$

This proves that, for  $\varepsilon$  sufficiently small, if  $\omega_\gamma \leq \varepsilon^{1+\alpha}$  for  $\gamma \in \Lambda_\alpha(\varepsilon)$ , then for all  $\theta \in \mathbb{T}_n^*$ ,  $H_{\omega,n,\theta}$  has an eigenvalue in  $[E_- - \varepsilon/2, E_- + \varepsilon/2]$ . Hence, we learn that

$$\mathbb{E}(N_{\omega,n}(E_- + \varepsilon/2) - N_{\omega,n}(E_- - \varepsilon/2)) \geq \frac{1}{C}n^{-d}P(\varepsilon, \alpha) \quad (2.7)$$

where  $P(\varepsilon, \alpha)$  is the probability of the event  $\{\omega; \forall \gamma \in \Lambda_\alpha(\varepsilon), \omega_\gamma \leq \varepsilon^{1+\alpha}\}$ . By assumption (HL2) and as  $n \sim \varepsilon^{-\nu}$ , we have

$$\liminf_{\substack{\varepsilon \rightarrow 0 \\ \varepsilon > 0}} \frac{\log |\log(P(\varepsilon, \alpha))|}{\log \varepsilon} \geq -\frac{d}{2}(1 + 3\alpha) - \alpha_- \quad (2.8)$$

where  $\alpha_-$  is defined in Theorem 0.2. Plugging (2.8) and (2.7) into (2.5), since  $\alpha > 0$  is arbitrary, this completes the proof of Theorem 0.2.  $\square$

### 3 An example where Lifshitz tails fail to exist

We now will construct an example of the type described in Theorem 0.5.

Let  $\varphi \in \mathcal{C}^\infty((-1/2, 1/2)^d)$  be positive, reflection symmetric, constant near the boundary of  $[-1/2, 1/2]^d$  and normalized in the cube. Let  $V = \Delta\varphi/\varphi$ ; it satisfies (0.2) and assumption (H2). Moreover,  $\varphi$  is the positive normalized ground state of  $-\Delta + V$  on  $[-1/2, 1/2]^d$  with Neumann boundary conditions, the associated ground state energy being 0. Let  $(\omega_\gamma)_{\gamma \in \mathbb{Z}^d}$  be Bernoulli random variables with support  $\{0, 1\}$ . Pick  $L \geq 1$  and let  $\varphi_L$  be the ground state of the operator  $H_{\omega,L}^N$  acting on  $[-1/2, L-1/2]^d$  with Neumann boundary conditions defined in Section 1. Then, this ground state can be described as follows

- in  $\gamma + [-1/2, 1/2]^d$ ,  $\varphi_L(\cdot) = L^{-d/2}\varphi(\cdot - \gamma)$  if  $\omega_\gamma = 1$ ;
- in  $\gamma + [-1/2, 1/2]^d$ ,  $\varphi_L(\cdot) = L^{-d/2}C_0$  if  $\omega_\gamma = 0$

where the constant  $C_0$  is chosen to be equal to the constant value of  $\varphi$  near the boundary of  $[-1/2, 1/2]^d$ . The thus constructed ground state is not normalized.

We can now use the results of [8] to obtain a lower bound on the second eigenvalue of  $H_{\omega,L}^N$  that is independent of  $\omega$ . Indeed, the construction above shows that, for any  $\omega$ ,

$$\begin{aligned} C_0 &\leq L^{d/2} \cdot \max_{x \in [-1/2, L-1/2]^d} \varphi_L(x) \leq \max_{x \in [-1/2, 1/2]^d} \varphi(x), \\ \min_{x \in [-1/2, 1/2]^d} \varphi(x) &\leq L^{d/2} \cdot \min_{x \in [-1/2, L-1/2]^d} \varphi_L(x) \leq C_0. \end{aligned}$$

Then, Theorem 1.4 of [8] applied to  $H_{\omega,L}^N$  and the Neumann Laplacian on the same cube guarantees that the second eigenvalue of  $H_{\omega,L}^N$  is larger than  $cL^{-2}$  where the constant  $c$  does not depend on  $\omega$ , nor on  $L$ . The standard upper bound for the integrated density of states by the normalized Neumann counting function (see e.g. [20, 5, 22]) yields, for any  $L \geq 1$ ,

$$N(E) \leq \mathbb{E} \left( \frac{\#\{\text{eigenvalues of } H_{\omega,L}^N \leq E\}}{L^d} \right).$$

If we pick  $L = (C^2 E)^{-1/2}$  for some  $C > 0$  such that  $cC^2 \geq 2$ , the second eigenvalue of  $H_{\omega,L}^N$  is larger than  $2E$ , this for any realization of  $\omega$ , we obtain

$$N(E) \leq L^{-d} = C^d E^{d/2}.$$

The standard lower bound for the integrated density of states by the normalized Dirichlet counting function (see e.g. [20, 5, 22]) yields

$$\mathbb{E} \left( \frac{\#\{\text{eigenvalues of } H_{\omega,L}^D \leq E\}}{L^d} \right) \leq N(E) \quad (3.1)$$

where  $H_{\omega,L}^D$  is the Dirichlet restriction of  $H_\omega$  to  $[-1/2, L - 1/2]^d$ . Let  $\psi_L$  be the (positive normalized) ground state of the Dirichlet Laplacian on  $[-1/2, L - 1/2]^d$ . And define  $\phi_L = \psi_L \cdot \varphi_L$ . This smooth function clearly satisfies Dirichlet boundary conditions and one computes, for some  $C > 1$ ,

$$\begin{aligned} \langle H_{\omega,L}^D \phi_L, \phi_L \rangle &= \langle (-\Delta + V_\omega) \phi_L, \phi_L \rangle \\ &= \langle (-\Delta \psi_L) \varphi_L, \phi_L \rangle - 2 \langle \nabla \psi_L \cdot \nabla \varphi_L, \phi_L \rangle \\ &= 2 \frac{\pi^2}{L^2} \|\phi_L\|^2 + \|\varphi_L |\nabla \psi_L|\|^2 \leq \frac{C}{L^2} \|\phi_L\|^2. \end{aligned} \quad (3.2)$$

where, in the final step, we integrated by parts and used the explicit form of the positive normalized ground state of the Dirichlet Laplacian.

Hence, taking  $L = CE^{-1/2}$ , as  $C > 1$ , (3.2) ensures that the ground state energy of  $H_{\omega,L}^D$  is less than  $E$ . Thus, from (3.1), we learn

$$\frac{1}{C^d} E^{d/2} = L^{-d} \leq N(E).$$

Finally, combining the two estimates, we have proved that, for the above random model, the density of states exhibits a van Hove singularity at the bottom of the spectrum that is, there exists  $C > 1$  such that, for  $E \geq 0$ , one has

$$\frac{1}{C} E^{d/2} \leq N(E) \leq C E^{d/2}.$$

This completes the proof of Theorem 0.5.  $\square$

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